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ON THE BRAUER GROUP OF A PROJECTIVE CURVE

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ABSTRACT
Using the torsion-theoretic description of the Brauer group of a projective variety, an elegant, direct description of the Brauer group of an arbitrary projective curve is established. It is given in terms of reflexive modules over the normalized curve and the Brauer groups of the occurring singularities.

Introduction

In [15] the authors established that for any regular projective variety X of dimension at most two we have $\mathrm{Br}(X) = \beta^g(R)$, where R is the homogeneous coordinate ring of X and $\beta^g(R)$ its so-called relative Brauer group, cf. [16]. This gives a module theoretic interpretation of $\mathrm{Br}(X)$. In the general case, i.e. X singular or of higher dimension, a direct description of $\mathrm{Br}(X)$ may still be given, but it is then of a torsion-theoretic nature. In this note, however, we show how $\mathrm{Br}(X)$ may be calculated in terms of relative Brauer groups β^g if one is willing to pass to the normalization $\tilde{X}$ of X at least for X of dimension one. The main technique used in this construction is the use of certain Mayer–Vietoris sequences for Brauer groups and the explicit calculation of the terms involved. These techniques of a cohomological nature are interesting in their own right, but we will not go into detail here as this would lead us too far. The material presented here is restricted to what is necessary; we have to refer to [15, 17] for basic definitions and terminology and to [13] for background on graded rings and modules. Let us just specify that all gradations are **Z**-gradations and that throughout R will denote a commutative positively graded ring with unit.

Let us start by gathering some general results and technicalities that play a role in the sequel. We will first derive some elementary results on the integral closure of graded domains.

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PROPOSITION 1. *Let S be a commutative graded domain and $\bar{S}$ the integral closure of S in its field of fractions, then the conductor c of $\bar{S}$ in S is a common graded ideal of S and $\bar{S}$.*

PROOF. By definition $c = \{x \in \bar{S};\ rx \in S \text{ for all } r \in \bar{S}\}$. If $x \in c$, let $x = x_{i_1} + \cdots + x_{i_n}$ be the homogeneous decomposition of x in $\bar{S}$; this is graded too (cf. [13]). If r is homogeneous in $\bar{S}$, then $rx = rx_1 + \cdots + rx_{i_n}$ with $rx \in S$ yields that $rx_{i_1}, \cdots, rx_{i_n} \in S$, i.e. $x_{i_1}, \cdots, x_{i_n} \in c$ is graded indeed. □

COROLLARY 2. *Let S be a* Gr-*local domain and let $\bar{S}$ be its integral closure. If c is the conductor of $\bar{S}$ in S, then S/c is* Gr-*local. If $\bar{S}$ is a finite S-module, then $\bar{S}$ is* Gr-*semilocal.*

PROOF. Recall from [13] that S Gr-local means that S possesses a unique Gr-maximal ideal (= maximal as a graded ideal). Now, by Proposition 1 we know that c is graded, hence contained in the unique graded maximal ideal M of S; consequently S/c is Gr-local. If $\bar{S}$ is a finite S-module, then there is only a finite number of prime ideals of $\bar{S}$ lying over M. These prime ideals are incomparable, so the graded ones amongst them will be Gr-maximal since M is. □

PROPOSITION 3. *Let R be a positively graded ring and consider a graded ring R' containing R such that R' is integral over R, then the canonical morphism $i : R \hookrightarrow R'$ induces a scheme morphism* $\mathrm{Proj}(R') \to \mathrm{Proj}(R)$.

PROOF. First note that R' has to be positively graded too, being integral over the positively graded ring R and graded itself. It is easily seen that we only have to verify the following: if a graded prime ideal P of R' contains R_+, then P also contains R'_+. Now, any $x \in h(R'_+)$ satisfies a relation $x^n = r_1x^{n-1} + \cdots + r_n$ for some $r_i \in h(R_+)$. So, if P contains R_+, then P contains x^n, i.e. P contains x. It follows that P contains $h(R'_+)$ hence also R'_+. □

From now on, assume R to be noetherian.

LEMMA 4. *Let c be the conductor of $\bar{R}$ in R. For any $f \in h(R_+)$ the following properties are valid:*

(4.1) *$Q_f^g(\bar{R})$ is the integral closure of $Q_f^g(R)$;*

(4.2) *the conductor d of $Q_f^g(\bar{R})$ in $Q_f^g(R)$ is exactly $Q_f^g(c)$.*

PROOF. Since Q_f^g is the localization functor obtained by inverting the (homogeneous) elements $\{1, f, f^2, \cdots\}$, it is clear that (4.1) follows from general localization results, cf. e.g. [2].

Since $Q_f^g(c) = Q_f^g(R)c$, it is easily verified that $Q_f^g(c) \subset d$.

Conversely, since R is noetherian, $\bar{R}$ is a finitely generated R-module, say $\bar{R} = Rx_1 + \cdots + Rx_n$, and we may choose the $x_1, \cdots, x_n$ in $h(\bar{R})$. If $yf^{-n} \in d$, then $yf^{-n}x_i \in Q_f^g(R)$ for each $i = 1, \cdots, x$. Hence $yx_i \in Q_f^g(R)$ and we may select $m \in \mathbf{N}$ large enough such that $f^m yx_i \in R$ for all i. Consequently $f^m y\bar{R} \subset R$ and $f^m y \in c$, i.e. $yf^{-n} \in Q_f^g(c)$. Thus $d = Q_f^g(c)$. □

COROLLARY 5. *Let R be a positively graded noetherian domain, then* $\mathrm{Proj}(\bar{R})$ *is the normalization of* $\mathrm{Proj}(R)$.

PROOF. This follows immediately from the local description of the normalization of a scheme in terms of an affine covering. □

PROPOSITION 6. *Let R be a positively graded noetherian domain which is generated as an R_0-algebra by R_1. Let $\bar{R}$ be the integral closure of R and c the conductor of $\bar{R}$ in R. For $P \in \mathrm{Proj}(R)$ the following statements are equivalent*:

(6.1) *the conductor c is contained in P;*

(6.2) *$Q_P^g(R)_0 = R_{(P)}$ is not integrally closed;*

(6.3) *$Q_P^g(R)$ is not integrally closed.*

PROOF. (1) ⇒ (2). Pick $f \in R_1 - P$, then $Q_f^g(c)_0 \subset Q_f^g(P)$ and thus the conductor c' of $Q_f^g(\bar{R})_0$ in $Q_f^g(R)_0$ is contained in $Q_f^g(P)_0$ because c' is the part of degree zero of the conductor of $Q_f^g(\bar{R})$ in $Q_f^g(R)$. Obviously, $Q_P^g(R) = Q^g_{Q_f^g(P)}(Q_f^g(R))$. But now $Q_f^g(R) = Q_f^g(R)_0[f, f^{-1}]$, hence the graded localization at $Q_f^g(P)$ may be obtained by inverting the elements of $Q_f^g(R)_0 - Q_f^g(P)_0$, so it follows that $Q_P^g(R)_0 = Q_{Q_f^g(P)_0}(Q_f^g(R)_0)$ and also that $Q_P^g(R)_0$ is not integrally closed.

(2) ⇒ (1), Again, choose $f \in R_1 - P$. Since $Q_P^g(R)_0$ is not integrally closed, $Q_f^g(c)_0 \subset Q_f^g(P)$ follows as in (1) ⇒ (2). From $Q_f^g(R) = Q_f^g(R)_0[f, f^{-1}]$, it then follows that $Q_f^g(P) \cap R = P$.

(2) ⇒ (3). We want to show that $R_{(P)}$ is integrally closed iff $Q_P^g(R)$ is integrally closed. Let $x \in h(Q_P^g(\bar{R}))$, say of degree $-s$, then $x^n + r_1 x^{n-1} + \cdots + r_n = 0$ for some $r_i \in Q_P^g(R)_{is}$. Pick $f \in Q_P^g(R)_1$ such that $Q_P^g(R) = R_{(P)}[f, f^{-1}]$, then we obtain a relation of the form $(f^s x) + (f^s r_1)(f^s x)^{n-1} + \cdots + (f^{ns} r_n) = 0$, all coefficients in $R_{(P)}$, hence $f^s x \in R_{(P)}$ and $x \in Q_P^g(R)$. The converse implication may be derived in a similar way. □

Let us now fix a field k and assume that R is a graded $k = R_0$-algebra generated by a finite number of elements of degree 1. Let $\bar{R}$ be the integral closure of R and let c be the conductor of $\bar{R}$ in R. Assume $X = \mathrm{Proj}(R)$ is a curve. We have already seen that $\tilde{X} = \mathrm{Proj}(\bar{R})$ is its normalization and $\tilde{c}$ its conductor sheaf $\mathrm{Ann}_X(\pi_* \mathbf{0}_{\tilde{X}} / \mathbf{0}_X)$, where $\pi : \tilde{X} \to X$ is the canonical covering. Let V resp. $\tilde{V}$ denote the closed subscheme of X resp. $\tilde{X}$ determined by c.

LEMMA 7. *With the above notations* $\mathrm{Pic}(\tilde{V}) = 0$.

PROOF. Indeed, by definition $\mathrm{Pic}(\tilde{V}) = H^1(\tilde{V}, \mathbf{0}^*_{\tilde{V}})$ and $\dim \tilde{V} = 0$ as $\tilde{V} \cong \mathrm{Proj}(\bar{R}/c)$, since $\bar{R}/c$ is Gr-semi-local by Corollary 2. It follows that $H^1(\tilde{V}, \mathbf{0}^*_{\tilde{V}}) = 0$, cf. [9]. □

THEOREM 8. *Under the above assumptions there is an exact sequence*

$$0 \to \mathrm{Br}\, X \to \mathrm{Br}\, \tilde{X} \oplus \mathrm{Br}\, V \to \mathrm{Br}\, \tilde{V}.$$

PROOF. Our assumptions imply that X may be covered by two open affines whose intersection is obviously affine too, X being separated. Similarly for the other schemes in the diagram

$$\begin{array}{ccc} \tilde{X} & \xleftarrow{\ i_1\ } & \tilde{V} \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ X & \xleftarrow[\ i\ ]{} & V \end{array}$$

But then, for the étale topology, there is an exact sequence of sheaves on X, cf. [10],

$$0 \to \mathbf{G}_{m,X} \to \pi_* \mathbf{G}_{m,\tilde{X}} \oplus i_* \mathbf{G}_{m,V} \to (\pi i_1)_* \mathbf{G}_{m,\tilde{V}} \to 0$$

which yields a long exact sequence

$$\cdots \to \mathrm{Pic}(\tilde{V}) \to H^2_{\acute{e}t}(X, \mathbf{G}_m) \to H^2_{\acute{e}t}(\tilde{X}, \mathbf{G}_m) \oplus H^2_{\acute{e}t}(V, \mathbf{G}_m) \to H^2_{\acute{e}t}(\tilde{V}, \mathbf{G}_m).$$

Now, using the fact that $\mathrm{Pic}(\tilde{V}) = 0$ by Lemma 7 and the fact that our assumptions imply $H_{\acute{e}t}(\ \ , \mathbf{G}_n)_t = \mathrm{Br}$, cf. [7, 10], we obtain the desired sequence.

If one wants to avoid the use of étale cohomology and the (deep) results of [7,10], a direct, elementary proof may be given as follows (we will only sketch it!). If S is an arbitrary positively graded ring, let us denote by $\mathbf{P}(S)$ resp. $\mathbf{AZ}(S)$ the category of locally projective sheaves of $\tilde{S}$-modules of finite type resp. of Azumaya Algebras on $\mathrm{Proj}(S)$. It is clear how the Cartesian diagram

$$\begin{array}{ccc} R & \longrightarrow & \bar{R} \\ \downarrow & & \downarrow \\ R/c & \longrightarrow & \bar{R}/c \end{array}$$

yields a commutative square

$$(*)\qquad \begin{array}{ccc} \mathbf{P}(R) & \longrightarrow & \mathbf{P}(\bar{R}) \\ \downarrow & & \downarrow \\ \mathbf{P}(R/c) & \longrightarrow & \mathbf{P}(\bar{R}/c) \end{array}$$

(use Proposition 3). Let $\mathbf{P}(\bar{R})\times_{\mathbf{P}(\bar{R}/c)}\mathbf{P}(R/c)$ be the associated Cartesian product, then passing to an affine covering and applying Milnor's theorem, cf. [1], one proves in a rather straightforward way that the canonical functor $\mathbf{P}(\bar{R})\times_{\mathbf{P}(\bar{R}/c)}\mathbf{P}(R/c)\to\mathbf{P}(R)$ is an isomorphism, i.e. that (*) is a Cartesian diagram. The analogous statement for **AZ** is valid too. But then, using Lemma 7, it suffices to mimic an elementary version of the proof given in [11] for the module theoretic analogue. Details are left to the reader. □

Recall from [16] that a graded R-algebra A is said to be a pseudo-Azumaya algebra if it is reflexive (i.e. A is finitely generated as an R-module and the canonical map $A\to A^{**}$ is an isomorphism of graded R-modules) and the map $(A\otimes A)^{**}\to \mathrm{END}_R(A)$ is an isomorphism. Here $\mathrm{END}_R(A)$ denotes the ring of graded R-linear endomorphisms of arbitrary degree. Two pseudo-Azumaya algebras A and B are said to be similar if we may find reflexive graded R-modules P and Q and an isomorphism of graded R-algebras $A\otimes_R \mathrm{END}_R(P)\tilde{\to} B\otimes_R \mathrm{END}_R(Q)$. The set of similarity classes of pseudo-Azumaya algebras may be endowed with a group structure in the obvious way and one thus obtains the so-called relative Brauer group $\beta^g(R)$ of R. For more details, cf. loc. cit.

Proposition 9. *Let* $\mathrm{Proj}(R)$ *be a connected normal projective curve, then* $\mathrm{Br}(\mathrm{Proj}(R))=\beta^g(\Gamma_*(\tilde{R}))$.

Proof. Cf. [15]. □

Let us now calculate $\mathrm{Br}(V)$ explicitly. First note that V contains only many graded prime ideals, say $V=\{P_1,\cdots,P_n\}$. Note that we do not distinguish whether we view a prime P as an element of $V_+(c)\subset\mathrm{Proj}(R)$ or as an element of $\mathrm{Proj}(R/c)$. We may find $f_1,\cdots,f_n\in h(R_+)$ such that $X_+(f_i)=\{P_i\}$. Indeed, just note that $\bigcap_{j\neq i}P_j\not\subset P_i$ since X is a curve, so there is a homogeneous $f_i\in\bigcap_{j\neq i}P_j-P_i$ and this f_i has the required property. It follows that V is the disjoint union of the affines $\mathrm{Spec}(Q^g_{f_i}(R/c)_0)=\mathrm{Spec}((Q^g_P(R)/Q^g_P(c))_0)$, so $\mathrm{Br}(V)=\bigoplus_{i=1}^n\mathrm{Br}((Q^g_P(R)/Q^g_P(c))_0)$. We claim that $[(Q^g_P(R)/Q^g_P(c))_0]_{\mathrm{red}}$ is just the function field $\mathbf{K}_X(P)$, for any $P\in V$. Indeed, first note that for any graded ring S we have $(S_0)_{\mathrm{red}}=(S_{\mathrm{red}})_0$, so we obtain

$$[(Q_P^g(R)/Q_P^g(c))_0]_{red} = [(Q_P^g(R)/Q_P^g(c))_{red}]_0 = [Q_P^g(R)/\mathrm{rad}(Q_P^g(c))]_0$$
$$= R_{(P)}/(\mathrm{rad}(Q_P^g(c)))_0.$$

Finally, since the primes in V are the only graded prime ideals of R (different from R_+) containing c and since the radical of a graded ideal is graded too, cf. [13], we may apply Proposition 1 to conclude that $\mathrm{rad}(Q_P^g(c)) = Q_P^g(P)$, and hence $[(Q_P^g(R)/Q_P^g(c))_0]_{red} = \mathbf{K}_X(P)$ indeed. We thus obtain

THEOREM 10. *Let $X = \mathrm{Proj}(R)$ be a connected projective curve; let $\bar{R}$ be the integral closure of R and let c be the conductor of $\bar{R}$ in R. Let V resp. $\tilde{V}$ be the closed subschemes of X resp. $\tilde{X} = \mathrm{Proj}(\bar{R})$ determined by c, then there is an exact sequence of abelian groups*

$$0 \to \mathrm{Br}(X) \to \beta^g(R) \oplus \bigoplus_{P \in V} \mathrm{Br}(\mathbf{K}_X(P)) \to \bigoplus_{Q \in \tilde{V}} \mathrm{Br}(\mathbf{K}_{\tilde{X}}(Q)).$$

PROOF. It suffices to apply the preceding remarks, taking into account the fact that $\mathrm{Br}(R) = \mathrm{Br}(R_{red})$ for any noetherian ring R, cf. [6]. □

EXAMPLE. Let $X = \mathrm{Proj}(\mathbf{R}[X, Y]/(X^2 + Y^2))$. The integral closure of R in this case is $R[X/y] \cong \mathbf{C}[y]$, which is regular, so $\beta^g\bar{R} = \mathrm{Br}^g\bar{R} = 0$ (cf. [17]). On the other hand, it is clear that $c = Ry$, since $x/y = x \in R$. Since $V = \{Ry\}$ and $R/c = \mathbf{R}[x]$ we have $\mathrm{Br}(\mathbf{K}_X(c)) = \mathbf{K}_X(c) = \mathrm{Br}(\mathbf{R}) = \mathbf{Z}/2\mathbf{Z}$ as unique factor. Finally $\bar{R}/c = \mathbf{C}[y]/(y) = \mathbf{C}$, so we obtain $\mathrm{Br}(X) = \mathbf{Z}/2\mathbf{Z}$. Other examples may be found in [17].

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